The (in)consistency of the bernoullian infinitesimals

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ABSTRACT *In this work we will deal with Peter Vickers' claim that Bernoulli's infinitesimals, as well as his whole version of calculus, are inconsistent. Following the same methodology that Vickers employs to evaluate, and reject, the charges of inconsistency against the first era calculus, we evaluate its own charge of inconsistency against Bernoulli's calculus. We will argue that the Postulate I of Die Differentialrechnung, the proposition in which Vickers focuses, is not an explicit contradiction and it does not lead us to one with contestable assumptions, therefore the formal theory of Bernoulli's calculus would not be necessarily inconsistent. We argue likewise that the relativity of the inconsistency to a logic and the lack of historical evidence also prevents the claim that the bernoullian justification of its calculus is inconsistent and thus, Vickers does not have enough elements to catalogue Bernoulli's calculus as inconsistent.*

1 Introduction

In this paper we deal with the question if infinitesimals as characterized by Johann Bernoulli in *Die Differentialrechnung* are inconsistent. Answering that inquiry is important not only because of the historiographical value of the case study, but also because of the impact that the supposed characterization of infinitesimals by Bernoulli both in the methodology as well as in the philosophy of science. On the one hand, some philosophers and logicians of science have suggested that inconsistencies are tolerated in scientific reasoning without the need of these to become a synonym of disaster; in particular, some of the defendants of such thesis have sustained as well that the interpretation of, at least, Bernoulli, Leibniz and Newton's calculus are inconsistent but not explosive (in the logical sense in which from a contradictions any other proposition follows logically; see for example [3]). On the other hand, a usual accusation against this tradition is to force the historical evidence to support their thesis about the tolerance to inconsistency through rational reconstructions that are unfaithful to the history of science. Therefore, we believe that answering this question will allow us to recognize important methodological criteria that need to be followed when analyzing similar inconsistencies to those attributed to Bernoulli, and at the same time it will allow us to support or weaken the theses related to the tolerance towards inconsistency.

Bernoulli's case is interesting because even Peter Vickers [8], probably the most meticulous debunker of supposed inconsistency cases in the history of science, claims that bernoullian infinitesimals are inconsistent and that Bernoulli was in fact compromised with their existence. Our goal here is to make a small contribution to a logically informed history and a philosophy of science, since we reply to the question about the inconsistency of bernoullian infinitesimals that they *may not be*, because consistency evaluations are relative to a logic that, even though is usually classical logic and that it was –at least in the order zero fragment- the dominant one in Bernoulli's time, is not the only one to make mathematics.

Note that we are not going to ask if Bernoulli was in fact commited to the existence of infinitesimals just as he defined them, if he was an instrumentalist with respect to them or if we have to resort to other parts of the bernoullian *corpus*, and even their interlocutors, to evaluate their properties, including their consistency. That would be a primarly historical, biographical investigation, whereas out question is about the logical properties of the so defined infinitesimals in the referred work. To approach other infinitesimals, whether they are contemporary –such as the diverse hyperreals studied from the work of Abraham Robinson or the nilpotents from the smooth infinitesimal analysis- or from Bernoulli's time –such as Nieuwenteijt's- is out of the scope of the present paper. We do not pretend to give a logical reconstruction of Bernoulli's calculus either but, following Vickers' methodology, to restrict our work to historically relevant propositions to evaluate the charge of inconsistency.

The work plan is as follows. In the first section we will explain Vickers' historically-eliminativist approach and reconstruct his argument to say that the first era calculus, with the exception of Bernoulli's, was not inconsistent. In the second section, following the same methodology, we will investigate if the logical reconstruction of Bernoulli's "paradoxical" claim, the Postulado I of *Die Differentialrechnung*, leads us to an explicit contradiction, as Vickers thinks and we argue that in fact it does not, unless one supposes that the adequate logic to reason about infinitesimals is such that $A \wedge \neg A$ follows. Finally, in the third section, we exploit the fact that of the relativity of the inconsistency to a logic to argue that the bernoullian justifications of its calculus does not allow to assert that its calculus is inconsistent either.

2 The inconsistency of the first era calculus

In "Was the early calculus an inconsistent theory?" ([7]) and afterwards in *Understanding Inconsistent Science*, [8, p. chapter 6], Peter Vickers has carefully examined the still well-known opinion that the first era calculus –that of Newton, Leibniz and their respective followers, whose techniques extended until the beginning of the XIX century- was inconsistent. But even after arguing in a pretty convincing manner that this was not so, Vickers sustains that Johann Bernoulli did have a compromise with infinitesimals with contradictory properties, for example, being identical and different from zero at the same time, and that he was probably the only one in this respect. The evidence that Vickers provides for that are "paradoxical" claims [8, p. 178] of Bernoulli such as the Postulate I of *Die Differentialrechnung*: "A quantity that increases or decreases infinitesimally does not increase nor decrease"*. And his defenses of the literalness of affirmations like this. Before pronouncing ourselves against Vickers' stance, we will expose the methodology he employs to evaluate the inconsistency claims.

* "Eine Grösse, die vermindert oder vermehrt wird um eine unendlich kleinere Grösse, wird weder vermindert noch vermehrt."

According to Vickers, in debates about the inconsistency of scientific theories, instead of taking care of defininf what a general theory is, and what “the first era calculus” is in this case, we should focus only in the historically relevant propositions to evaluate the charges of inconsistency. That is what Vickers calls “theoretical eliminativism” [8, chapter 2]. In this case the first era calculus, these propositions could be found in two levels, this is, the leven of the *formal theory* and the level of *justification*, this is, the level that explains which are the basical conceptual assumptions, as well as how and why this works or should work.

Vickers’ argument to assert that the first era calculus is not inconsistent is as follows. If the calculus is inconsistent, then either the formal theory is inconsistent or the justification given in the formal theory is successful and inconsistent. But the formal theory of the first era calculus is basically the following algorithm [8, p. 150]:

1. Write the equation in the form $y = f(x)$.
2. Calculate $\frac{f(x + o) - f(x)}{0}$ and simplify.
3. Remove any term multiple of *o*.
4. The resulting term then is the derivate.†

If this algorithm is inconsistent it is because it contains self-contradictory rules (such as “Do not follow this rule”), or because it provides inconsistent results with some already established mathematical truths, or because it provides different results using the same procedure. But none of this happened with the algorithm aforementioned. For example, those who pretended to have obtained some inconsistent result with some established mathematical truth incorrectly applied the algorithm at some point. This way the formal theory is not inconsistent, so in order for the first era calculus to be inconsistent, some of the successful justifications of the algorithm would have to be so.

† Or, in a single sentence: if we calculate $\frac{f(x + o) - f(x)}{0}$ for any function $y = f(x)$ y where *o* is a numerical constant and it is eliminated by any remaining term that is multiple of *o*, then we have the derivate. [8, p. 154]

For Vickers, a successful justification of the first era calculus is a justification of the steps of the algorithm, and its relations with other areas of the already established mathematics, accepted by the pertinent community as the true one or, at least, as "candidate for truth". However, according to Vickers, many of the justifications offered were not even considered successful, but merely good heuristics or "useful fictions" – for example, many of the ones who resorted to infinitesimals. Others were not even justifications, but mere expositions of the algorithm; lastly there were justifications that were actually accepted by different mathematicians as "candidates for truth" (the explanations in terms of fluxions or other cinematics, or the ones that appealed to infinitesimals but as variable quantities), but were not successful in the sense that they were not accepted by the whole pertinent community since they did not convincingly explained some part of the algorithm or of its connections with other areas of the already established mathematics. Nevertheless, not every failed explanation is inconsistent: it might as well be incomplete or incoherent.

This way, if nor the logically reconstructed theory nor the successful justifications were inconsistent, there were no reasons to state that the first era calculus was inconsistent. The only exception for Vickers is Johann Bernoulli, since he "stands out as making the strongest explicit commitment to a set of inconsistent assumptions" [8, p. 147]; "he [Johann Bernoulli] reasoned as if he believed in self-contradictory infinitesimals because he *did* believe in self-contradictory infinitesimals." [8, p. 183] y "Bernoulli, in believing two inconsistent conjuncts at slightly different times, is committed to believing the full conjunction at a single time." [8, p. 185], with which "[h]ere we have what is surely one of the most radical positions in the history of science and mathematics: Bernoulli seems quite clearly to embrace an obvious contradiction." [8, p. 178] However, following his own methodology, Vickers would have to show us that either the logical reconstruction of Bernoulli's theory or his justification for the theory are non-existent. In the following section we will see that the logical reconstruction of bernoullian infinitesimals is not necessarily inconsistent.

3 Formalizing *Postulate I*

Vickers [8, p. 185] considers a way in which supposedly contradictory bernoullian infinitesimals do not trivialize the theory by restricting the validity of some logical principles. To do so, at the beginning of a derivation the step from

The infinitesimal equals zero and different from zero.

to

By conjunction elimination, the infinitesimal is different from zero.

And at the end of a derivation, the valid step would be

The infinitesimal equals zero and is different from zero.

Then, by conjunction elimination, the infinitesimal equals zero.

Vickers thinks this is problematic, because not only does it require that the logic was paraconsistent to avoid trivializing the theory from the initial assumption, that the infinitesimal equals zero and is different from zero, but that it would require a very complicated logic to allow only certain conjunction eliminations in determined steps of a derivation.

We only wish to point that this is not so wild to use the steps Vickers mentionsas part of the algorithm of the calculus. In relevance logics, there are conjunctions that work as fusions and that have a more restrict conjunction elimination:

A and B; A implies that B implies C

C; by conjunction elimination (cf. [5])

Then, an appropriate strict conditional, such as “A implies that $\neg A$ implies $\neg A$”, would help justify the elimination on the first step and, *mutatis mutandis*, something similar would happen for the second.

However, there are logical options that are less complex to understand Bernoulli’s infinitesimals that do not even require for them to be considered as inconsistent. Let us recall Bernoulli’s Posulate I: “A quantity, which is diminished or increased by an infinitely small

quantity, is neither diminished nor increased”. But, is this a contradiction? Note that the postulate does not say

For every number x and every infinitesimal ϵ, ϵx and not ϵx.

since the sentence clearly has a conditional nature: if the quantity increases (or decreases) infinitesimally, it does not increase (nor decrease). Thus, another lecture could be

For every number *x* and every infinitesimal ϵ, if ϵx then not ϵx.

which does not even make sense, because one does not deny terms, such as the result of operations like product, and the first paraphrases given also has this problem: one does not conjunct terms.

Another paraphrases would be:

For every number *x* and every infinitesimal ϵ, $(\epsilon x = x)$.

This is not a contradiction, and it seems to comply with the spirit of Bernoulli's postulate: a quantity *x* that varies infinitesimally (ϵx) does not vary, it is equal to itself (x). However, we said before that the sentence has a conditional nature, that would not be exposed in the paraphrases we just gave. This can be solved as follows:

For every number *x*, every number *y* and every infinitesimal ϵ, if $(\epsilon x = x)$ then $(x = y)$.

This is almost the *microcancellation principle* of the smooth infinitesimal analysis (cf. [1, chapters 1, 2, 7 and 8]) we would only need to have written 'ϵx = ex' instead of 'ϵx = y'. Given that Bernoulli's postulate implies the microcancellation principle –just instanciate 'y' with 'ϵy'-, that is also incompatible with classical logic, in particular with $A \vee \neg A$.

Therefore, if the historically relevant proposition to evaluate the inconsistency in the case of Bernoulli is Postulate I, the formal theory of bernoullian calculus is not necessarily inconsistent: it is if the logic used to reason about infinitesimals has to be one in which $A \vee \neg A$ is valid.

4 The not inconsistency of the bernoullian infinitesimals

Qualifying bernoullian infinitesimals as inconsistent require more resources than those allowed by the historicist-eliminativist methodology that Vickers self-imposes. Vickers seems to assume that the only historically relevant proposition in the case of Bernoulli is Postulate I. We can concede that. Immediately after, Vickers should have showed us that either the formal theory of Bernoulli's calculus is inconsistent, or that its justification of the same is inconsistent.

But, paying further attention in the formalization of Postulate I, we observed that it is not an explicit contradiction. An important lesson of that was showed in the previous section is that, just as it was emphasized by recent authors (cf. for example [6]): the consistency is relative to a given logic. Without assuming that $A \vee \neg A$ is valid, Postulate I is not necessarily inconsistent, in the sense that it does not necessarily lead us to an explicit contradiction. What is left for us is to investigate if the bernoullian justification of calculus is inconsistent.

Let us assume that Bernoulli has been one of those who "leaves logic for afterwards" (cf. [4, p. 9]) to his use of calculus and its justification. Without the intuitionistic logic as a resource, it is most likely that Bernoulli's justifications were either incomplete, by not being able to explain how could Postulate I be calif without reaching false results such as q=0, or incoherent, by supporting claims as "Both $A \vee \neg A$ and Postulate I are valid in my infinitesimal calculus and I do not reach false results such as 1=0" without noticing and accepting the contradiction, but according to Vickers this happened with many of the first era analysts. In fact, at the end of the chapter [8, p. 189], Vickers seems to recognize that that there is not an agreement between Bernoulli's practice and the justification he would have given of the calculus, but this would show, if anything, that Bernoulli's justification is inadequate, not necessarily inconsistent. However, a more detailed study of Bernoulli's writings would be in place to obtain more certainty. Thus, Vickers has not given enough elements to claim that Bernoulli's calculus is inconsistent.

5 Conclusions

In this paper we wondered if, as Vickers claims, Bernoulli's infinitesimals are inconsistent and, with that, his whole version of calculus. We showed the methodology that Vickers employs to evaluate the charges of inconsistency against the first era calculus and we adopted it to evaluate its own inconsistency charge against Bernoulli's calculus. We found that the proposition that Vickers judges as historically relevant, Postulate I of *Die Differentialrechnung*, is not an explicit contradiction nor does it lead us to one without certain assumptions, in particular, that the logic that needs to be used to reason with infinitesimals is one in which $A \vee \neg A$ is valid, which is why the formal theory of Bernoulli's calculus would not necessarily be inconsistent. The relativity of the inconsistency to the logic and the lack of historical evidence also prevents the claim that the bernoullian justification of its calculus is inconsistent, which is why Vickers does not have enough elements to catalogue Bernoulli as a special case, insofar as inconsistencies, between the first era analysts.[‡]

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